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HOW TO FIND DETERMINANTS BY USING EXPONENTIAL GENERATING FUNCTIONS

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ABSTRACT

As we know, let alone to find the determinant of infinite matrix, it is difficult to find the determinant of some n x n matrixes by the usual methods like, the cofactor method and Crammer's rule. But now we will show how to find the determinant of some n x n matrices and how to find the determinant of some infinite matrix by using Exponential Generating Functions. In this paper we will consider matrices having 1, 2, 3, 4, 5…on the supper diagonal, 0's on the upper and identical entries on each diagonal below the supper diagonal. Here we will try how to obtain the determinant of n x n upper left corner sub matrix of a given **infinite matrix** by introducing Exponential Generating functions of some sequences and how to get a sequence by calculating the determinant of n x n upper left corner sub matrix of infinite matrix. we will also check the correctness of the determinant by using Numerical method.

KEYWORDS: Infinite Matrix; Determinant of Matrices; Exponential Generating Functions; Sequences; Sub Matrix

1. INTRODUCTION

To understand the whole work, it is better to know about a matrix, determinants, Generating functions and some sequences. So we will discuss these terms before the actual work.

What are Generating Functions?

One of the main tasks in combinatory is to develop tools for counting. Perhaps, one of the most powerful tools frequently used in counting is the notion of Generating functions. Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series. Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints, and the number of ways to make change for a dollar using coins of different denominations(Discrete Mathematics and Its Applications, Seventh Edition, Kenneth H. Rosen Monmouth University (and formerly AT&T Laboratories page 537).

A generating function is a continuous function associated with a given sequence. For this reason, generating functions are very useful in analyzing discrete problems involving sequences of numbers or sequences of functions. (Generating Functions and Their Applications Agustinus Peter Sahanggamu MIT Mathematics Department Class of 2007 18.104 Term Paper Fall 2006 page 1)

In mathematics a Generating function is a formal power series whose coefficients encode information about a sequence $\{a_n\}$ that is indexed by the natural number n. Generating functions can be used to solve determinants of some nxn and then an infinite matrix by relating the terms of the sequence for which we get a generating function to the determinant of an upper left corner nxn matrix of an infinite matrix.. Even though there are various types of Generating functions, in this paper, we introduce the idea of Exponential generating Functions.

Definition

The exponential generating function A(x) of a sequence $\{a_n\}^{\infty}{}_{n}$ is defined as

A(X) = $\sum_{r=0}^{\infty} a_r \frac{x^r}{r!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots \; a_r \frac{x^r}{r!} \dots$ the exponential generating functions have the following property.

Lemma

If A(x) is the exponential generating function of a sequence $\{a_n\}_{n=0}^{\infty}$ then

$\sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!} = A'(x)$

Proof: Differentiate both sides of the definition of A(x) in

A(X) = $\sum_{nr=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots \; a_r \frac{x^r}{r!} \dots$

$A'(x) = \sum_{m=1}^{\infty} m a_m \frac{x^{m-1}}{m!} = \sum_{m=1}^{\infty} a_m \frac{x^{m-1}}{(m-1)!} = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}$

Some Examples of Generating Functions of Some Sequences

i) $\langle 1,1,1,\dots \rangle \leftrightarrow 1 + x + x^2 + x^3 + \dots = \sum_{n-0}^{\infty} x^n = \frac{1}{1-x}$ is the ordinary Generating function

ii) $\langle 1,1,\frac{1}{2!},\frac{1}{3!},\frac{1}{4!},\dots \rangle \leftrightarrow \sum_{r=0}^{\infty} \frac{x^r}{r!} = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ is its exponential Generating function.

iii) $\langle$ 1, 2, 3, 4... $\rangle$ $\leftrightarrow$ $1+2x+3x^2+4x^3+\dots = \frac{1}{(1-x)^2}$ is the generating function for counting numbers.

iv) The generating function for the sequence (1, k, k^2, k^3...), where k is an ordinary constant is $1+kx+k^2x^2+k^3x^3+\dots = \frac{1}{1-kx}$

v) $\langle$ 1,-1, 1,-1... $\rangle$ $\leftrightarrow 1 - x + x^2 - x^3 + \dots = \frac{1}{1-(-x)} = \frac{1}{1+x}$

vi) $\langle$ 1, a, a^2, a^3... $\rangle$ $\leftrightarrow$ $1 + ax + a^2 x^2 + \dots = \frac{1}{1-ax}$

vii) $\langle$ 1, 0, 1, 0, 1... $\rangle \leftrightarrow$ $1+ x^2 + x^4 + x^6 + ... = \frac{1}{1-x^2}$

1.2 OPERATIONS ON GENERATING FUNCTIONS

Let $A(x) = a_0+a_1 x+a_2 x^2 + a_3 x^3+...$ and $B(x) = b_0+b_1 x+b_2 x^2 + b_3 x^3+...$ be the generating functions for the sequences (a_r) and (b_r) respectively, then

1.2.1 Product Rule (Convolution)

Although multiplying two functions is as natural as adding them, as it is seen below, general term of the resulting sequence of multiplication is not the product of i th terms of the sequences. In fact: -

$A(x)\times B(x) = a_0b_0 + (a_0b_1+ a_1b_0)x +(a_0b_2+ a_1b_1 +a_2b_0)x^2....$is the generating function for the sequence (c_r), where

$c_r= a_0b_0 + a_1b_{r-1}+.... a_{r-1} b_1+ a_rb_0$, r = 0,1,2,3,...

Proof

To evaluate the product $a(x) \times B(x)$ let us use the following table.

Table 1: Product Table

$\times$	b_0	b_1x	$b_2 x^2$	$b_3 x^3 ...$
a_0	a_0b_0	a_0b_1x	$a_0b_2 x^2$	$a_0b_3x^3 ..$
a_1x	a_1b_0x	$a_1b_1x^2$	$a_1b_2x^3$	$a_1b_3x^4$
$a_2 x^2$	$a_2b_0 x^2$	$a_2b_1x^3$	$a_2b_2x^4$	$a_2b_3x^5$
$a_3 x^3$	$a_3b_0x^3$	$a_3b_1x^4$	$a_3b_2x^5$	$a_3b_3x^6 ...$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

If we follow the arrow, we get the required product

E.g. If $A(x) \leftrightarrow \langle$ 1,2,2,2, $\rangle \leftrightarrow \frac{1+x}{1-x}$ and $B(x) \leftrightarrow \langle$ 1, 1, 1,... $\rangle \leftrightarrow \frac{1}{1-x}$ then

$$A(x).\ B(x) = \left(\frac{1+x}{1-x}\right)\left(\frac{1}{1-x}\right) = \frac{1+x}{(1-x)^2} \leftrightarrow \langle 1,3,5,7,9.... \rangle$$

Using the product rule we have the following:

(1-x) A(x) is the generating function for the sequence (c_r) where

$c_0 = a_0$ and $c_r= a_r - a_{r-1}$ for all $r \geq 1$ and

$\frac{A(x)}{1-x}$ is the generating function for the sequence (c_r) where

$c_r = a_0 + a_1 + a_2 + \ldots + a_r$ for all r.

Remark

since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots..$ and

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^n}{n!} + \ldots..$$

Then we have $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \cos x$ and

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sin x$$

1.2.2 Addition Rule

$A(x)+B(x)$ is the generating function for the sequence (c_r) where $c_r = a_r + b_r$, r = 0,1,2,3...

Proof

$A(x)+B(x) = (a_0+a_1x+a_2 x^2 + a_3 x^3+\ldots) + (b_0+b_1x+b_2x^2 + b_3 x^3+\ldots)$

$= (a_0+b_0) + (a_1+b_1)x+(a_2+b_2)x^2 +(a_3+b_3.)x^3+\ldots$

$= c_0+ c_1x + c_2x^2+ c_3x^3+\ldots$

where $c_r = a_r + b_r$, r = 0,1,2,3...

eg. if $A(x) = \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots \leftrightarrow \langle\ 1,2,4,8,\ldots\ \rangle$

$B(x) = \frac{3}{1-3x} = 3\left(\frac{3}{1-3x}\right) = 3(1 + 3x + 9x^2 + 27x^3 + \cdots) = 3 + 9x + 27x^2 + 81x^3 + \cdots \leftrightarrow \langle\ 3,9,27,\ldots\ \rangle$

then $A(x) + B(x) = \frac{1}{1-2x} + \frac{3}{1-3x} = \frac{4-9x}{1-5x+6x^2} \leftrightarrow \langle\ 4, 11, 31\ldots\ \rangle$

1.2.3 Constant /Scaling/ Rule

$\langle ca_0, ca_1, ca_2, ca_{3\ldots}\ ca_{n}\ldots \rangle \leftrightarrow cA(x)$.

Proof

$\langle\ ca_0, ca_1, ca_2, ca_{3\ldots}\ \rangle \leftrightarrow ca_0 + ca_1x+ca_2 x^2+ca_3x^3 +\ldots$

$=c(a_0+a_1x+a_2x^2+a_3 x^3+\ldots \rangle = cA(x)$

E.g. If $\langle\ 1,2,3,4,\ldots \rangle \leftrightarrow 1+2x+3x^2 + 4x^3+\ldots = \frac{1}{(1-x)^2}$ and c=2, we have

$\frac{2}{(1-x)^2} = 2 + 4x + 6x^2 + 8x^3 + \ldots \leftrightarrow (2,4,6,8,10, \ldots)$

1.2.4 THE DERIVATIVE RULE

If $\langle a_0, a_1, a_2 .. \rangle \leftrightarrow A(x)$, then $\langle a_1, 2a_2, 3a_3, \ldots \rangle \leftrightarrow A'(x)$

Proof

$\langle a_0, a_1, a_2, \rangle \leftrightarrow A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$\Rightarrow \frac{d}{dx} A(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots = A'(x) \leftrightarrow \langle a_1, 2a_2, 3a_3{}^{2,} \ldots \rangle$

E.g. $\langle 1,1,1,\ldots. \rangle \leftrightarrow \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = A(x)$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots \leftrightarrow \langle 1, 2, 3, 4, \ldots \rangle = A'(x)$

And $x A'(x) \leftrightarrow \langle 0, 1, 2, 3, \ldots \rangle \leftrightarrow \frac{x}{(1-x)^2}$

Hence $[xA`(x)]` = \left(\frac{x}{(1-x)^2}\right)' = \frac{(1+x)}{(1+x)^3} \leftrightarrow \langle 1, 4, 9, 16, \ldots \rangle$ (square number Sequence)

Note:$\lfloor x^n \rfloor$ Given a generating function A(x) we use$\lfloor x^n \rfloor$ A(x) to denote a_n, the coefficient of x^n. (270 Chapter 10 Ordinary Generating Functions)

MATRIX AND DETERMINANTS

Definition

A matrix is a rectangular array of mn quanties $a_{ij} \begin{pmatrix} i = 1,2,3,\ldots., m \,\& \\ j = 1,2,3,\ldots, n \end{pmatrix}$ in m - rows and n- columns. It is called an m $\times$n matrix or a matrix of order m $\times$ n and read as m by n matrix. The numbers a_{ij} are called the elements (constituents or coordinates or entries) of the matrix and we will denote the matrix by $\{a_{ij}\}$ or A. The suffix ij of an element a_{ij} indicates that it occurs in the i^{th} row and j^{th} column. when n=m we call this a square matrix. For a square matrix nxn, if n$\to \infty$ then the matrix is called an infinite matrix.

In Explicit form $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & a_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & a_{nn} \end{pmatrix}$ rows

Columns

NOTE

1) If A = $\{a_{ij}\}_{m \times n}$ and B= $\{b_{ij}\}_{n \times p}$, and AB =C, then C = $\{c_{ij}\}_{m \times p}$

where $c_{ij} = \sum_{k=1}^{n} a_{ij} b_{kj}$

2) For any matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & a_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & a_{nn} \end{pmatrix}$$

The supper diagonal

The main diagonal

Sub diagonal

MATRIX MUITIPLLCATION

Two matrixes A and B are conformable for the product AB when the number of columns in A is equal to the number of rows in B. If A is an m x n matrix and B is an n x p matrix then their product AB is defined as m x p matrix whose (ij) th element is obtained by multiplying the element of the ith row of A in the corresponding elements of the jth column of B and summing the products so obtained. So the (ij) th element of the product AB is the inner product of the ith row of A and the j th column of B.

DETERMINANTS

Definition

Determinant of a matrix A is a specific real number assigned to A It is denoted by det (A) or $|A|$ Or for n $\geq$ 1 the determinant of an n x n matrix A= (a_{ij}) along the first row is the sum of n-terms of the form $\pm$ a_{ij} det A_{ij} with plus and minus signs alternating where the entries $a_{11}, a_{12}, a_{13} ... a_{1n}$ are from the first row of A.

In symbols, det A = a_{11} det A_{11} - a_{12} det A_{12} +... + $(-1)^{n+1}$ a_{1n} det A_{1n} = $\sum_{j=1}^{n}$ $(-1)^{1+j}$ a_{1j} A_{1j} and det A_{1j} is the determinant of the sub matrix which is obtained by removing the 1st row and the jth column.

Actually det A_{1j} is called the minor of a_{1j} and $(-1)^{1+j}$ a_{1j} det A_{1j} is called the cofactor of $a_{ij.}$

CRAMER'S RULE

Let A be an invertible n x n matrix. For any b in R^n the unique solution x of

Ax = b has entries given by $x_i = \frac{\det A_i(b)}{\det A}$ where i = 1,2, 3,... n and A_i (b) is the matrix obtained from A by replacing column i by the vector b.

Proof

Denote the column of A by $a_1, a_2, a_3, \ldots a_n.$ and the column of the n x n identity matrix I by $e_1, e_2, e_3, \ldots e_n.$

If Ax =b then the definition of matrix multiplication shows that

$A I_i(x) = A [e_1, e_2, e_3, \ldots x \ldots e_n] = [Ae_1, Ae_2 \ldots Ax \ldots Ae_n]$

$= [a_1, a_2, \ldots b \ldots a_n] = A_i(b)$

by the multiplicative property of determinants

$(\det A) \det I_i(x) = \det A_i(b)$

$\Rightarrow (\det A) X_i = \det A_i(b)$

$\Rightarrow X_i = \frac{\det A_i(b)}{\det A}$

Determinant of a matrix can be obtained by the cofactor method or by using the Cramer's rule. But now we are Interested to show how to find the determinant of several matrices by using Generating functions.

The matrices whose determinants we will be evaluating have all 1's on the super diagonal, 0's above the supper diagonal ; and identical entries on each diagonal below the supper diagonal, perhaps with the exception of the first column,

2. DESCRIPTION OF THE METHOD

In this topic we will see how to get a sequence from the given matrix by calculating the determinant of each upper left nxn square matrices of the given matrix. All matrices in this section will have 1, 2, 3, 4, 5, on the supper, 0`s above and identical entries on each diagonal below, perhaps with the exception of the first column. Hence we will equate the nth term of the sequence with determinant of each upper left nxn square matrices of the given matrix. We begin with a typical example as follows

Example: $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 3 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 1 & 0 & 4 & 0 & 0 & . & . & . \\ 1 & 1 & 1 & 1 & 0 & 5 & 0 & . & . & . \\ 1 & 1 & 1 & 1 & 1 & 0 & 6 & . & . & . \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & . & . & . \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}$ (1)

Consider the System Below

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 2x & 0 & 0 & 0 & 0 & . & . & . \\ x^2 & 0 & 3x^2 & 0 & 0 & 0 & . & . & . \\ x^3 & x^3 & 0 & 4x^3 & 0 & 0 & . & . & . \\ x^4 & x^4 & x^4 & 0 & 5x^4 & 0 & . & . & . \\ x^5 & x^5 & x^5 & x^5 & 0 & 6x^5 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} 0 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ . \\ . \\ . \end{pmatrix} \quad *$$

Let $F(x) = 1+f_1x+f_2x^2+f_3x^3+\ldots$ and $C'(x) = x+x^2+x^3+\ldots$ be ordinary generating function.

$f_1=0$

$2f_2x=x$

$f_1x^2+3f_3x^2=x^2$

$f_1x^3+f_2x^3+4f_4x^3=x^3$

$f_1x^4+f_2x^4+f_3x^4+5f_5x^4=x^4$, *

Summing on both sides of (*) we get a first order differential equation

$F'(x) + f_1(x^2+x^3+x^4+\ldots) + f_2(x^3+x^4+x^5+\ldots) + \ldots = x+x^2+x^3+x^4+x^5+\ldots = C'(x)$

$F'(x) + f_1x(x+x^2+x^3+x^4+\ldots) + f_2x^2(x+x^2+x^3+x^4+x^5+\ldots) + f_3x^3(x+x^2+x^3+x^4+x^5+\ldots)\ldots = C'(x)$

$F'(x) + (f_1x + f_2x^2 + f_3x^3+\ldots\ldots)(x+x^2+x^3+x^4+x^5+\ldots) = C'(x)$

$F'(x) + (f_1x + f_2x^2 + f_3x^3+\ldots\ldots)\,C'(x) = C'(x)$

$F'(x)+(F(x)-1)C'(x)=C'(x) \Rightarrow F'(x)= -(F(x)-1)C'(x)+C'(x)=(1-F(x)+1)C'(x) =(2-F(x)C'(x)\ \frac{F'(x)}{2-F(x)} = C'(x)$ Solving the differential equation with F(0)=1 and C(0)=0

we have $F(x) = 2 - e^{-C(x)}$ **

Now $C'(x) = \frac{x}{1-x} \Rightarrow C(x) = -x - ln(1-x)$.substituting this in ** we have

$F(x) = 2 - e^x + xe^x$

But we know that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$

$$\Rightarrow F(x) = 2 - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... \right) + x(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...)$$

$$\Rightarrow F(x) = 1 - \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... \right) + x + x^2 + \frac{x^3}{2!} + ... + \frac{x^n}{(n-1)!} + ...)$$

$=1 + \sum_{n=1}^{\infty}(\frac{1}{(n-1)!} - \frac{1}{n!})x^n = 1 + \sum_{n=1}^{\infty}(\frac{n-1}{(n)!})x^n$ hence $f_n = \frac{n-1}{n!}$

Now we want to evaluate the upper left corner determinants as follows

$$|0| = 0, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = 2, \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -3, \dots \dots \dots$$

where the n[th] such determinant will be denoted by $D_{n,n} \geq 1$

One way to determine say D_5 is as follows, consider the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 1 & 0 & 4 & 0 \\ 1 & 1 & 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad (2)$$

where the right hand side is the first column from the original matrix of (1) for n=5

By crammer's rule and properties of determinants, we have

$$f_5 = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 & 1 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 1 & 0 & 4 & 0 \\ 1 & 1 & 1 & 0 & 5 \end{vmatrix}} = \frac{4}{5!} = ((-1)^{n-1}\frac{D_5}{5!}$$

.In general by induction we have $n!f_n = (-1)^{n-1} D_n$

i.e we have $D_n = (-1)^{n-1} n!\, f_n = (-1)^{n-1}(n-1)$ for all n>0

Proposition: suppose

$$D = \begin{pmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ b_2 & b_1 & 2 & 0 & 0 & 0 & 0 & . & . & . \\ b_3 & b_2 & b_1 & 3 & 0 & 0 & 0 & . & . & . \\ b_4 & b_3 & b_2 & b_1 & 4 & 0 & 0 & . & . & . \\ b_5 & b_4 & b_3 & b_2 & b_1 & 5 & 0 & . & . & . \\ b_6 & b_5 & b_4 & b_3 & b_2 & b_1 & 6 & . & . & . \\ b_7 & b_6 & b_5 & b_4 & b_3 & b_2 & b_1 & . & . & . \\ b_8 & b_7 & b_6 & b_5 & b_4 & b_3 & b_2 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}.$$

**

Let B' (x) = $\sum_{n=1}^{\infty} b_n x^{n-1}$ and let F(x) =1+ $\sum_{n=1}^{\infty} f_n x^n$ be the generating function for some number such that F' (x) = (2-F(x)) B`(x).Then

$$D_n = \begin{cases} (-1)^{n-1} f_n \ if \ F(x) is \ an \ EGF \\ (-1)^{n-1} n! \, f_n \ if \ F(x) is \ an \ OGF \end{cases}$$

Proof: If

$$D = \begin{pmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ b_2 & b_1 & 2 & 0 & 0 & 0 & 0 & . & . & . \\ b_3 & b_2 & b_1 & 3 & 0 & 0 & 0 & . & . & . \\ b_4 & b_3 & b_2 & b_1 & 4 & 0 & 0 & . & . & . \\ b_5 & b_4 & b_3 & b_2 & b_1 & 5 & 0 & . & . & . \\ b_6 & b_5 & b_4 & b_3 & b_2 & b_1 & 6 & . & . & . \\ b_7 & b_6 & b_5 & b_4 & b_3 & b_2 & b_1 & . & . & . \\ b_8 & b_7 & b_6 & b_5 & b_4 & b_3 & b_2 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}.$$

$D_1=b_1$, $D_2=\begin{vmatrix} b_1 & 1 \\ b_2 & b_1 \end{vmatrix} = b_1^2 - b_2$, $D_3 = \begin{vmatrix} b_1 & 1 & 0 \\ b_2 & b_1 & 2 \\ b_3 & b_2 & b_1 \end{vmatrix} = b_1^3 - 3b_1b_2 + 2b_3 \ldots$

If F(x) is an *E.G.F* of f_1, f_2, then consider the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ b_1x & 2x & 0 & 0 & 0 & . & . \\ b_2x^2 & b_1x^2 & 3x^2 & 0 & 0 & . & . \\ b_3x^3 & b_2x^3 & b_1x^3 & 4x^3 & 0 & . & . \\ b_4x^4 & b_3x^4 & b_2x^4 & b_1x^4 & & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} f_1 \\ f_2/2! \\ f_3/3! \\ f_4/4! \\ f_5/5! \\ . \\ . \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ . \\ . \end{pmatrix}$$

****

Then $f_1 + 2\frac{f_2}{2!}x + 3\frac{f_3}{3}x^2 + \cdots + f_1(b_1x + b_2x^2 + \cdots) + \frac{f_2}{2!}(b_1x^2 + b_2x^3 + \cdots..) = C^{'}(x)$

$F^{'}(x) + f_1xC^{'}(x) + \frac{f_2}{2!}x^2C^{'}(x) + \frac{f_3}{3!}x^3C^{'}(x) + \cdots = C^{'}(x)$

$F^{'}(x) + (f_1x + \frac{f_2}{2!}x^2 + \frac{f_3}{3!}x^3)C^{'}(x) + \cdots = C^{'}(x)$

$F^{'}(x) + (F(x) - 1)C^{'}(x) = C^{'}(x)$

$F^{'}(x) = \left(2 - F(x)\right)C^{'}(x)$

And by using Cramer's rule on ***** we have

$$f_1 = b_1 = D_1, \frac{f_2}{2!} = \frac{\begin{vmatrix} 1 & b_1 \\ b_1 & b_2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ b_1 & 2 \end{vmatrix}} = \frac{b_2 - {b_1}^2}{2} \Rightarrow f_2 = -D_2$$

$$\frac{f_3}{3!} = \frac{\begin{vmatrix} 1 & 0 & b_1 \\ b_1 & 2 & b_2 \\ b_2 & b_1 & b_3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ b_1 & 2 & 0 \\ b_2 & b_1 & 3 \end{vmatrix}} = \frac{\begin{vmatrix} 2 & b_2 \\ b_1 & b_3 \end{vmatrix} + b_1\begin{vmatrix} b_1 & 2 \\ b_2 & b_1 \end{vmatrix}}{-6} = \frac{2b_3 - 3b_1b_2 + {b_1}^3}{-6} = \frac{D_3}{3!}$$

$\Rightarrow f_3 = D_3$

Continuing in this way we have $D_n = (-1)^{n-1}f_n$

Example 1

Consider 1,2,4,8,16,32,64, ... $= f_0, f_1, f_2, f_3, \ldots$ which has an O.G.F

$F(x) = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + 64x^6 + \cdots = \frac{1}{1-2x}$

$F^{'}(x) = \frac{2}{(1-2x)^2}$ And $2 - F(x) = \frac{1-4x}{1-2x}$ hence $B^{'}(x) = \frac{F'(x)}{2-F(x)} = \left(\frac{2}{1-2x}\right)\left(\frac{1}{1-4x}\right)$

$\Rightarrow B^{'}(x) = 2 + 12x + 56x^2 + 176x^3 + 992x^4 + \cdots$

Hence $b_1 = 2, b_2 = 12, b_3 = 56, b_4 = 176, b_5 = 992, \ldots$

Now if

$$D = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 12 & 2 & 2 & 0 & 0 & 0 & 0 & . & . & . \\ 56 & 12 & 2 & 3 & 0 & 0 & 0 & . & . & . \\ 176 & 56 & 12 & 2 & 4 & 0 & 0 & . & . & . \\ 992 & 176 & 56 & 12 & 2 & 5 & 0 & . & . & . \\ . & 992 & 176 & 56 & 12 & 2 & 6 & . & . & . \\ . & . & 992 & 176 & 56 & 12 & 2 & . & . & . \\ . & . & . & 992 & 176 & 56 & 12 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}.$$

$then\ D_n = (-1)^{n-1} n!\, f_n = (-1)^{n-1} 2^n n!\ for\ all\ n \geq 1$

Example 2: Let $F(x) = \frac{1}{1-4x}$ be an ordinary Generating Function for a certain sequence.

$\Rightarrow F'(x) = \frac{4}{(1-4x)^2}$

$\Rightarrow C'(x) = \frac{F'(x)}{2-F(x)} = \frac{\frac{4}{(1-4x)^2}}{2-(\frac{1}{1-4x})} = \frac{\frac{4}{(1-4x)^2}}{\frac{2-8x-1}{1-4x})} = \left(\frac{4}{1-4x}\right)\left(\frac{1}{1-8x}\right)$

$=4(1 + 4x + 16x^2 + 64x^3 + 256x^4 + 1024x^5 + 4096x^6 + \cdots)(1 + 8x + 64x^2 + \cdots)$

$=(4 + 16x + 64x^2 + 256x^3 + 1024x^4 + 4096x^5 + \cdots)(1 + 8x + 64x^2 + 512x^3 + \cdots)$

$==(4 + 48x + 448x^2 + 3840x^3 + \cdots$ by product rule.

Where $b_1 = 4, b_2 = 48, b_3 = 448, b_4 = 3840, \ldots$

Now if

$$D = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 48 & 4 & 2 & 0 & 0 & 0 & 0 & . & . & . \\ 448 & 48 & 4 & 3 & 0 & 0 & 0 & . & . & . \\ 3840 & 448 & 48 & 4 & 4 & 0 & 0 & . & . & . \\ . & 3840 & 448 & 48 & 4 & 5 & 0 & . & . & . \\ . & . & 3840 & 448 & 48 & 4 & 6 & . & . & . \\ . & . & . & 3840 & 448 & 48 & 4 & . & . & . \\ . & . & . & . & 3840 & 448 & 48 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}.$$

$then\ D_n = (-1)^{n-1} n!\, f_n = (-1)^{n-1} 4^n n!\ for\ all\ n > 0$

Conflict of Interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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